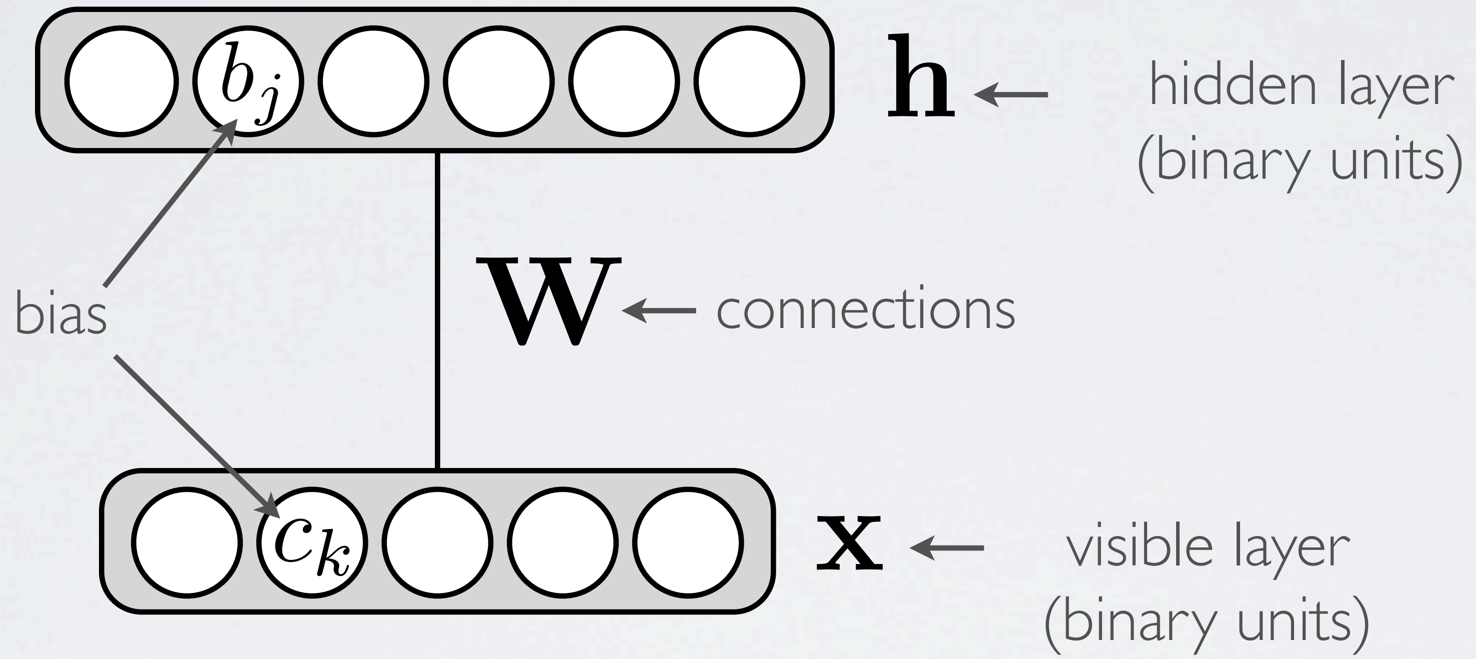
Neural networks

Restricted Boltzmann machine - inference

RESTRICTED BOLTZMANN MACHINE

Topics: RBM, visible layer, hidden layer, energy function



Energy function:

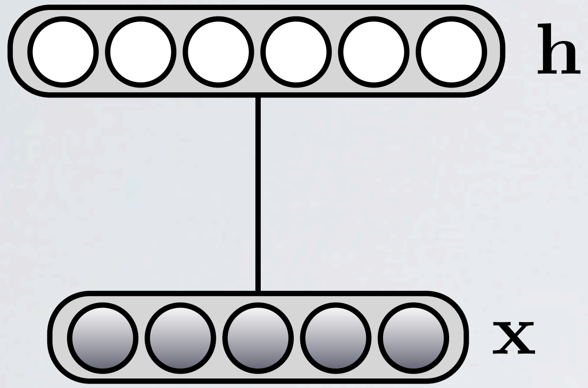
$$
\begin{aligned}
E(\mathbf{x}, \mathbf{h}) &= -\mathbf{h}^\top \mathbf{W} \mathbf{x} - \mathbf{c}^\top \mathbf{x} - \mathbf{b}^\top \mathbf{h} \\
&= -\sum_j \sum_k W_{j,k} h_j x_k - \sum_k c_k x_k - \sum_j b_j h_j
\end{aligned}
$$

Distribution: $p(\mathbf{x}, \mathbf{h}) = \exp(-E(\mathbf{x}, \mathbf{h}))/Z$

partition function
(intractable)

INFERENCE

Topics: conditional distributions

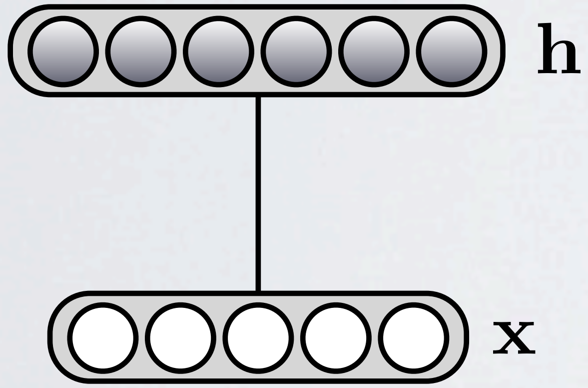


$$p(\mathbf{h}|\mathbf{x}) = \prod_j p(h_j|\mathbf{x})$$

$$p(h_j = 1|\mathbf{x}) = \frac{1}{1 + \exp(-(b_j + \mathbf{W}_{j\cdot}\mathbf{x}))}$$

$$= \mathrm{sigm}(b_j + \mathbf{W}_{j\cdot}\mathbf{x})$$

j th row of $\mathbf{W}$



$$p(\mathbf{x}|\mathbf{h}) = \prod_k p(x_k|\mathbf{h})$$

$$p(x_k = 1|\mathbf{h}) = \frac{1}{1 + \exp(-(c_k + \mathbf{h}^\top \mathbf{W}_{\cdot k}))}$$

k th column of $\mathbf{W}$

$$= \mathrm{sigm}(c_k + \mathbf{h}^\top \mathbf{W}_{\cdot k})$$

$p(\mathbf{h}|\mathbf{x})$

$$p(\mathbf{h}|\mathbf{x}) \quad = \quad p(\mathbf{x},\mathbf{h}) / \sum_{\mathbf{h}'} p(\mathbf{x},\mathbf{h}')$$

$$\begin{aligned} p(\mathbf{h}|\mathbf{x}) &= p(\mathbf{x},\mathbf{h}) / \sum_{\mathbf{h}'} p(\mathbf{x},\mathbf{h}') \\ &= \frac{\exp(\mathbf{h}^\top \mathbf{W}\mathbf{x} + \mathbf{c}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{h})/Z}{\sum_{\mathbf{h}' \in \{0,1\}^H} \exp(\mathbf{h}'^\top \mathbf{W}\mathbf{x} + \mathbf{c}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{h}')/Z} \end{aligned}$$

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\end{aligned}
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&= \frac{\exp(\sum_j h_j \mathbf{W}_{j\cdot}\mathbf{x} + b_j h_j)}{\sum_{h'_1 \in \{0,1\}} \cdots \sum_{h'_H \in \{0,1\}} \exp(\sum_j h'_j \mathbf{W}_{j\cdot}\mathbf{x} + b_j h'_j)}
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&= \frac{\prod_j \exp(h_j \mathbf{W}_{j\cdot}\mathbf{x} + b_j h_j)}{\sum_{h_1' \in \{0,1\}} \cdots \sum_{h_H' \in \{0,1\}} \prod_j \exp(h_j' \mathbf{W}_{j\cdot}\mathbf{x} + b_j h_j')}
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&= \frac{\prod_j \exp(h_j \mathbf{W}_{j\cdot}\mathbf{x} + b_j h_j)}{\left(\sum_{h'_1 \in \{0,1\}} \exp(h'_1 \mathbf{W}_{1\cdot}\mathbf{x} + b_1 h'_1)\right) \dots \left(\sum_{h'_H \in \{0,1\}} \exp(h'_H \mathbf{W}_{H\cdot}\mathbf{x} + b_H h'_H)\right)}
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&= \frac{\prod_j \exp(h_j \mathbf{W}_{j\cdot}\mathbf{x} + b_j h_j)}{\left(\sum_{h'_1 \in \{0,1\}} \exp(h'_1 \mathbf{W}_{1\cdot}\mathbf{x} + b_1 h'_1)\right) \dots \left(\sum_{h'_H \in \{0,1\}} \exp(h'_H \mathbf{W}_{H\cdot}\mathbf{x} + b_H h'_H)\right)} \\
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&= \frac{\prod_j \exp(h_j \mathbf{W}_{j\cdot}\mathbf{x} + b_j h_j)}{\prod_j (1 + \exp(b_j + \mathbf{W}_{j\cdot}\mathbf{x}))}
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&= \frac{\prod_j \exp(h_j \mathbf{W}_{j \cdot} \mathbf{x} + b_j h_j)}{\prod_j (1 + \exp(b_j + \mathbf{W}_{j \cdot} \mathbf{x}))} \\
&= \prod_j \frac{\exp(h_j \mathbf{W}_{j \cdot} \mathbf{x} + b_j h_j)}{1 + \exp(b_j + \mathbf{W}_{j \cdot} \mathbf{x})}
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&= \frac{\prod_j \exp(h_j \mathbf{W}_{j\cdot}\mathbf{x} + b_j h_j)}{\left(\sum_{h'_1 \in \{0,1\}} \exp(h'_1 \mathbf{W}_{1\cdot}\mathbf{x} + b_1 h'_1)\right) \cdots \left(\sum_{h'_H \in \{0,1\}} \exp(h'_H \mathbf{W}_{H\cdot}\mathbf{x} + b_H h'_H)\right)} \\
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&= \frac{\prod_j \exp(h_j \mathbf{W}_{j\cdot}\mathbf{x} + b_j h_j)}{\prod_j (1 + \exp(b_j + \mathbf{W}_{j\cdot}\mathbf{x}))} \\
&= \prod_j \frac{\exp(h_j \mathbf{W}_{j\cdot}\mathbf{x} + b_j h_j)}{1 + \exp(b_j + \mathbf{W}_{j\cdot}\mathbf{x})} \\
&= \prod_j p(h_j|\mathbf{x})
\end{aligned}
$$

$p(h_j = 1|\mathbf{x})$

$$p(h_j = 1|\mathbf{x}) \quad = \quad \frac{\exp(b_j + \mathbf{W}_{j\cdot}\mathbf{x})}{1 + \exp(b_j + \mathbf{W}_{j\cdot}\mathbf{x})}$$

$$
\begin{aligned}
p(h_j = 1|\mathbf{x}) &= \frac{\exp(b_j + \mathbf{W}_{j\cdot}\mathbf{x})}{1 + \exp(b_j + \mathbf{W}_{j\cdot}\mathbf{x})} \\
&= \frac{1}{1 + \exp(-b_j - \mathbf{W}_{j\cdot}\mathbf{x})}
\end{aligned}
$$

$$
\begin{aligned}
p(h_j = 1|\mathbf{x}) &= \frac{\exp(b_j + \mathbf{W}_{j\cdot}\mathbf{x})}{1 + \exp(b_j + \mathbf{W}_{j\cdot}\mathbf{x})} \\
&= \frac{1}{1 + \exp(-b_j - \mathbf{W}_{j\cdot}\mathbf{x})} \\
&= \mathrm{sigm}(b_j + \mathbf{W}_{j\cdot}\mathbf{x})
\end{aligned}
$$

LOCAL MARKOV PROPERTY

Topics: local Markov property

- In general, we have the following property:

$$
\begin{aligned}
p(z_i|z_1,\ldots,z_V) &= p(z_i|\mathrm{Ne}(z_i)) \\
&= \frac{p(z_i,\mathrm{Ne}(z_i))}{\sum_{z_i'} p(z_i',\mathrm{Ne}(z_i))} \\
&= \frac{\prod_{\substack{f \text{ involving } z_i \\ \text{and any } \mathrm{Ne}(z_i)}} \Psi_f(z_i,\mathrm{Ne}(z_i))}{\sum_{z_i'} \prod_{\substack{f \text{ involving } z_i \\ \text{and any } \mathrm{Ne}(z_i)}} \Psi_f(z_i',\mathrm{Ne}(z_i))}
\end{aligned}
$$

  - z_i is any variable in the Markov network (x_k or h_j in an RBM)
  - $\mathrm{Ne}(z_i)$ are the neighbors of z_i in the Markov network