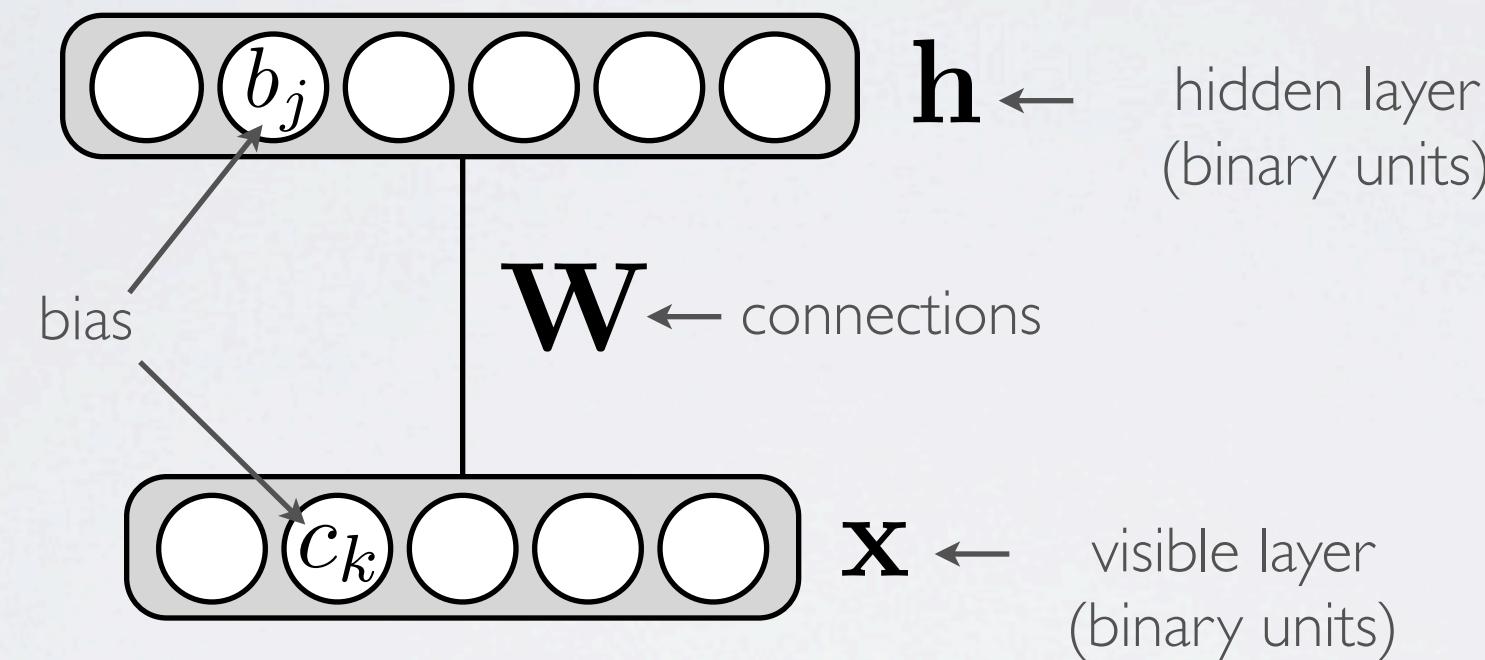


Neural networks

Restricted Boltzmann machine - free energy

RESTRICTED BOLTZMANN MACHINE

Topics: RBM, visible layer, hidden layer, energy function



$$\begin{aligned} \text{Energy function: } E(\mathbf{x}, \mathbf{h}) &= -\mathbf{h}^\top \mathbf{W} \mathbf{x} - \mathbf{c}^\top \mathbf{x} - \mathbf{b}^\top \mathbf{h} \\ &= -\sum_j \sum_k W_{j,k} h_j x_k - \sum_k c_k x_k - \sum_j b_j h_j \end{aligned}$$

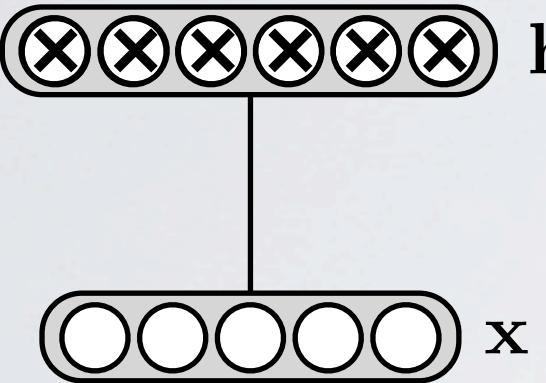
$$\text{Distribution: } p(\mathbf{x}, \mathbf{h}) = \exp(-E(\mathbf{x}, \mathbf{h}))/Z$$

partition function
(intractable)

FREE ENERGY

Topics: free energy

- What about $p(\mathbf{x})$?



$$\begin{aligned}
 p(\mathbf{x}) &= \sum_{\mathbf{h} \in \{0,1\}^H} p(\mathbf{x}, \mathbf{h}) = \sum_{\mathbf{h} \in \{0,1\}^H} \exp(-E(\mathbf{x}, \mathbf{h}))/Z \\
 &= \exp \left(\mathbf{c}^\top \mathbf{x} + \sum_{j=1}^H \log(1 + \exp(b_j + \mathbf{W}_{j \cdot} \mathbf{x})) \right) / Z \\
 &= \exp(-F(\mathbf{x}))/Z
 \end{aligned}$$

free energy

$$p(\mathbf{x})$$

$$p(\mathbf{x}) = \sum_{\mathbf{h} \in \{0,1\}^H} \exp(\mathbf{h}^\top \mathbf{W} \mathbf{x} + \mathbf{c}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{h}) / Z$$

$$\begin{aligned}
p(\mathbf{x}) &= \sum_{\mathbf{h} \in \{0,1\}^H} \exp(\mathbf{h}^\top \mathbf{W} \mathbf{x} + \mathbf{c}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{h}) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) \sum_{h_1 \in \{0,1\}} \cdots \sum_{h_H \in \{0,1\}} \exp \left(\sum_j h_j \mathbf{W}_{j \cdot} \mathbf{x} + b_j h_j \right) / Z
\end{aligned}$$

$$\begin{aligned}
p(\mathbf{x}) &= \sum_{\mathbf{h} \in \{0,1\}^H} \exp(\mathbf{h}^\top \mathbf{W} \mathbf{x} + \mathbf{c}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{h}) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) \sum_{h_1 \in \{0,1\}} \cdots \sum_{h_H \in \{0,1\}} \exp \left(\sum_j h_j \mathbf{W}_{j \cdot} \mathbf{x} + b_j h_j \right) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) \left(\sum_{h_1 \in \{0,1\}} \exp(h_1 \mathbf{W}_{1 \cdot} \mathbf{x} + b_1 h_1) \right) \cdots \left(\sum_{h_H \in \{0,1\}} \exp(h_H \mathbf{W}_{H \cdot} \mathbf{x} + b_H h_H) \right) / Z
\end{aligned}$$

$$\begin{aligned}
p(\mathbf{x}) &= \sum_{\mathbf{h} \in \{0,1\}^H} \exp(\mathbf{h}^\top \mathbf{W}\mathbf{x} + \mathbf{c}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{h}) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) \sum_{h_1 \in \{0,1\}} \cdots \sum_{h_H \in \{0,1\}} \exp \left(\sum_j h_j \mathbf{W}_{j \cdot} \mathbf{x} + b_j h_j \right) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) \left(\sum_{h_1 \in \{0,1\}} \exp(h_1 \mathbf{W}_{1 \cdot} \mathbf{x} + b_1 h_1) \right) \cdots \left(\sum_{h_H \in \{0,1\}} \exp(h_H \mathbf{W}_{H \cdot} \mathbf{x} + b_H h_H) \right) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) (1 + \exp(b_1 + \mathbf{W}_{1 \cdot} \mathbf{x})) \cdots (1 + \exp(b_H + \mathbf{W}_{H \cdot} \mathbf{x})) / Z
\end{aligned}$$

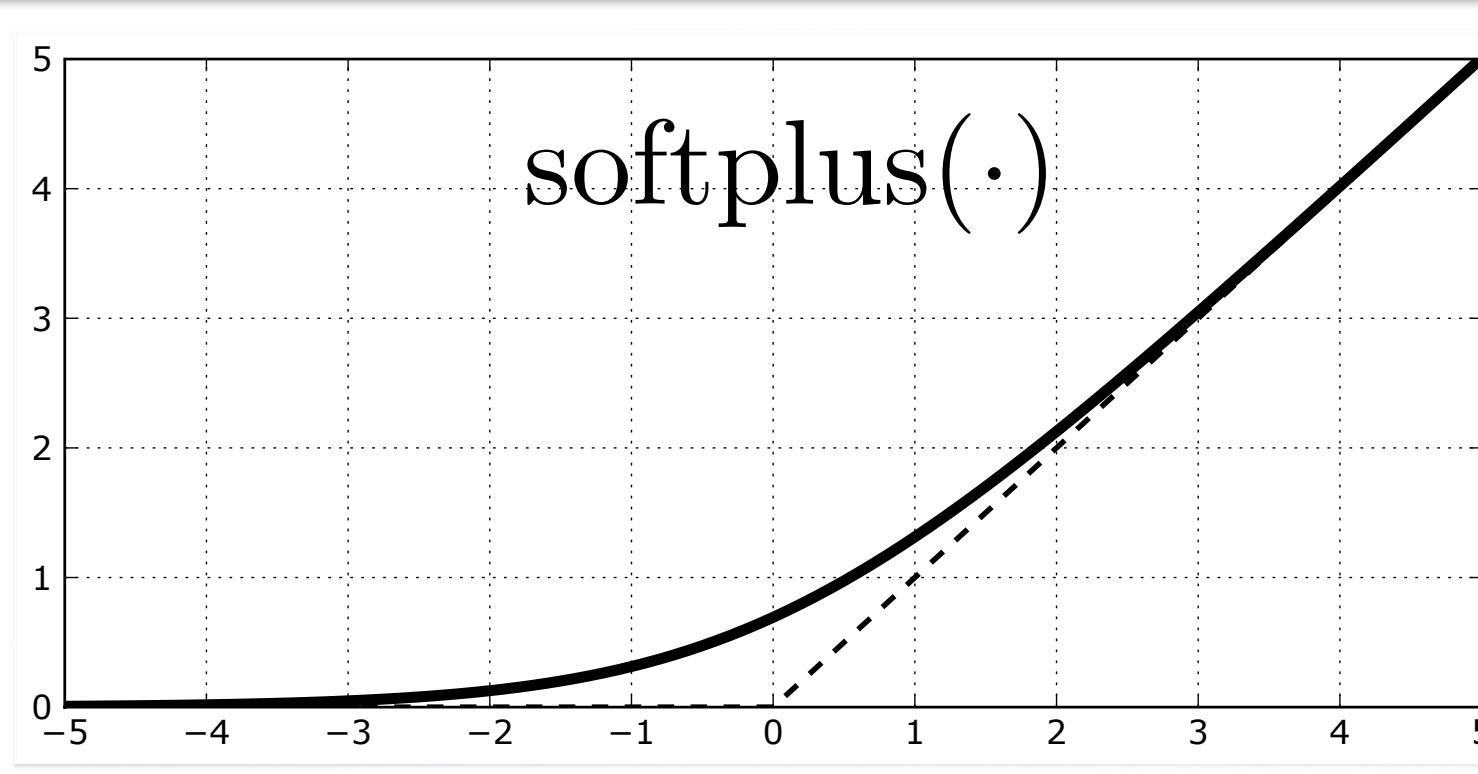
$$\begin{aligned}
p(\mathbf{x}) &= \sum_{\mathbf{h} \in \{0,1\}^H} \exp(\mathbf{h}^\top \mathbf{W}\mathbf{x} + \mathbf{c}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{h}) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) \sum_{h_1 \in \{0,1\}} \dots \sum_{h_H \in \{0,1\}} \exp \left(\sum_j h_j \mathbf{W}_{j \cdot} \mathbf{x} + b_j h_j \right) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) \left(\sum_{h_1 \in \{0,1\}} \exp(h_1 \mathbf{W}_{1 \cdot} \mathbf{x} + b_1 h_1) \right) \dots \left(\sum_{h_H \in \{0,1\}} \exp(h_H \mathbf{W}_{H \cdot} \mathbf{x} + b_H h_H) \right) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) (1 + \exp(b_1 + \mathbf{W}_{1 \cdot} \mathbf{x})) \dots (1 + \exp(b_H + \mathbf{W}_{H \cdot} \mathbf{x})) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) \exp(\log(1 + \exp(b_1 + \mathbf{W}_{1 \cdot} \mathbf{x}))) \dots \exp(\log(1 + \exp(b_H + \mathbf{W}_{H \cdot} \mathbf{x}))) / Z
\end{aligned}$$

$$\begin{aligned}
p(\mathbf{x}) &= \sum_{\mathbf{h} \in \{0,1\}^H} \exp(\mathbf{h}^\top \mathbf{W}\mathbf{x} + \mathbf{c}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{h}) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) \sum_{h_1 \in \{0,1\}} \cdots \sum_{h_H \in \{0,1\}} \exp \left(\sum_j h_j \mathbf{W}_{j \cdot} \mathbf{x} + b_j h_j \right) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) \left(\sum_{h_1 \in \{0,1\}} \exp(h_1 \mathbf{W}_{1 \cdot} \mathbf{x} + b_1 h_1) \right) \cdots \left(\sum_{h_H \in \{0,1\}} \exp(h_H \mathbf{W}_{H \cdot} \mathbf{x} + b_H h_H) \right) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) (1 + \exp(b_1 + \mathbf{W}_{1 \cdot} \mathbf{x})) \cdots (1 + \exp(b_H + \mathbf{W}_{H \cdot} \mathbf{x})) / Z \\
&= \exp(\mathbf{c}^\top \mathbf{x}) \exp(\log(1 + \exp(b_1 + \mathbf{W}_{1 \cdot} \mathbf{x}))) \cdots \exp(\log(1 + \exp(b_H + \mathbf{W}_{H \cdot} \mathbf{x}))) / Z \\
&= \exp \left(\mathbf{c}^\top \mathbf{x} + \sum_{j=1}^H \log(1 + \exp(b_j + \mathbf{W}_{j \cdot} \mathbf{x})) \right) / Z
\end{aligned}$$

RESTRICTED BOLTZMANN MACHINE

Topics: free energy

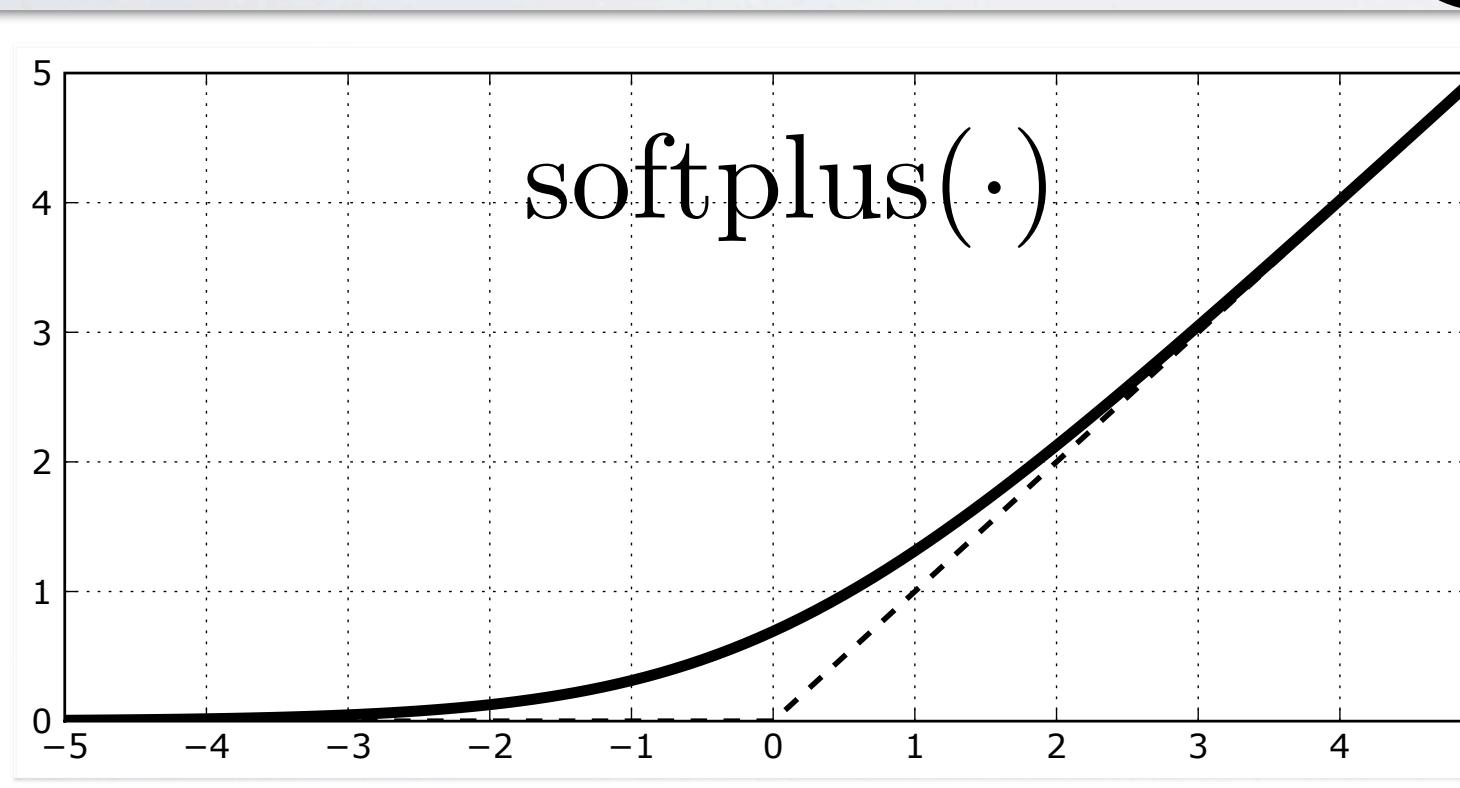
$$\begin{aligned}
 \text{Diagram: } & \text{A Restricted Boltzmann Machine (RBM) diagram showing two layers of nodes. The top layer, labeled 'h', has 6 nodes, each containing an 'X'. The bottom layer, labeled 'x', has 5 nodes, each containing a circle. A vertical line connects the two layers.} \\
 p(\mathbf{x}) &= \exp \left(\mathbf{c}^\top \mathbf{x} + \sum_{j=1}^H \log(1 + \exp(b_j + \mathbf{W}_{j \cdot} \mathbf{x})) \right) / Z \\
 &= \exp \left(\mathbf{c}^\top \mathbf{x} + \sum_{j=1}^H \text{softplus}(b_j + \mathbf{W}_{j \cdot} \mathbf{x}) \right) / Z
 \end{aligned}$$



RESTRICTED BOLTZMANN MACHINE

Topics: free energy

$$\begin{aligned}
 \text{Diagram: } & \text{A Restricted Boltzmann Machine (RBM) diagram showing two layers of nodes. The top layer, labeled 'h', has 6 nodes, 3 of which are filled with 'x' marks. The bottom layer, labeled 'x', has 5 nodes, all of which are empty circles. A vertical line connects the two layers.} \\
 p(\mathbf{x}) &= \exp \left(\mathbf{c}^\top \mathbf{x} + \sum_{j=1}^H \log(1 + \exp(b_j + \mathbf{W}_{j \cdot} \mathbf{x})) \right) / Z \\
 &= \exp \left(\mathbf{c}^\top \mathbf{x} + \sum_{j=1}^H \text{softplus}(b_j + \mathbf{W}_{j \cdot} \mathbf{x}) \right) / Z
 \end{aligned}$$

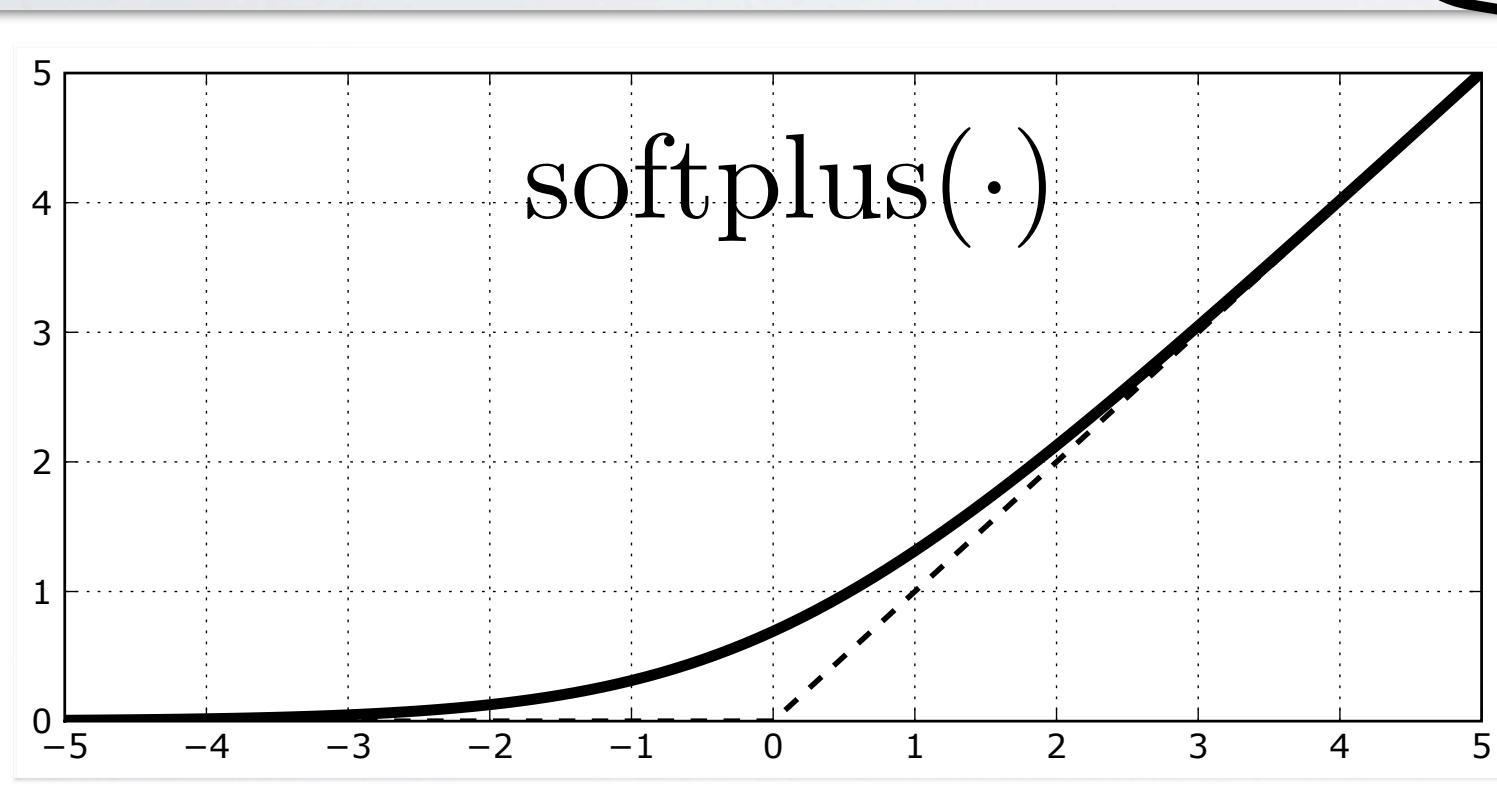


bias the prob of each x_i

RESTRICTED BOLTZMANN MACHINE

Topics: free energy

$$\begin{aligned}
 \text{h} \quad p(\mathbf{x}) &= \exp \left(\mathbf{c}^\top \mathbf{x} + \sum_{j=1}^H \log(1 + \exp(b_j + \mathbf{W}_{j \cdot} \mathbf{x})) \right) / Z \\
 &= \exp \left(\mathbf{c}^\top \mathbf{x} + \sum_{j=1}^H \text{softplus}(b_j + \mathbf{W}_{j \cdot} \mathbf{x}) \right) / Z
 \end{aligned}$$



“feature” expected in \mathbf{x}
bias the prob of each x_i

RESTRICTED BOLTZMANN MACHINE

Topics: free energy

$$\begin{aligned}
 \text{Diagram: } & \text{A Restricted Boltzmann Machine (RBM) diagram showing two layers of nodes. The top layer, labeled 'h', has 6 nodes, 3 of which are crossed out (X). The bottom layer, labeled 'x', has 5 nodes. A vertical line connects the two layers.} \\
 \text{Equation 1: } & p(\mathbf{x}) = \exp \left(\mathbf{c}^\top \mathbf{x} + \sum_{j=1}^H \log(1 + \exp(b_j + \mathbf{W}_{j \cdot} \mathbf{x})) \right) / Z \\
 \text{Equation 2: } & = \exp \left(\mathbf{c}^\top \mathbf{x} + \sum_{j=1}^H \text{softplus}(b_j + \mathbf{W}_{j \cdot} \mathbf{x}) \right) / Z
 \end{aligned}$$

Annotations pointing to the second equation:

- “feature” expected in \mathbf{x}
- bias of each feature
- bias the prob of each x_i

